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THE VARIATIONAL INEQUALITIES

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Abstract—We are discussing here two variational inequalities.

1. INTRODUCTION

Let B be a real Banach space (not necessarily reflexive), B^* its topological conjugate space endowed with weak* topology, (u,v) the pairing between $u \in B$ and $v \in B^*$. Let K be a subset of B and let T be an operator from K into B^*. A set K is said to be a *cone* if $\lambda x \in K$ for all $x \in K$ and all $\lambda \geq 0$. The set $K^\circ = \{v \in B^* \mid (u,v) \leq 1, \forall u \in K\}$ is called the *polar* of K. In the case that K is a cone, K° is also a cone and $K^\circ = \{v \in B^* \mid (u,v) \leq 0, \forall u \in K\}$.

The *variational inequality*, VI(T,K), associated with T and K is the problem to find $\bar{x} \in K$ such that

$$(x - \bar{x}, T\bar{x}) \geq 0, \qquad \forall x \in K. \tag{1.1}$$

This problem has been investigated by many authors, see, for example, the survey paper of Moré [1] and the references therein. We remark that Hartman and Stampacchia have obtained an existence theorem for VI(T,K) under the conditions that T is continuous and K is compact convex in the finite dimensional case [2, Lemma 3.1]. Also, using Fan's well-known minimax inequality theorem [3], Théra [4, Theorem 2] gave another existence theorem for VI(T,K) under more general settings.

In Section 2, by using a well-known result of Fan [5], we give an extension of Théra's theorem (see Theorem 2.1) and its applications. In particular, we obtain conditions under which an operator is surjective. Also, some necessary and sufficient conditions for the existence of solutions to VI(T,K) are given. In Section 3, we prove some existence theorems for VI(T,K) for pseudomonotone operator T.

A closely related problem to VI(T,K) is the so called complementarity problem. Let K be a cone in B. The *complementarity problem*, CP, is to find $\bar{x} \in K$ such that

$$T\bar{x} \in -K^\circ \quad \text{and} \quad (\bar{x}, T\bar{x}) = 0. \tag{1.2}$$

Such problems were introduced by Karamardian [6] and have been extensively studied in the literature. See, e.g., [1,6–17] and the references therein. Also, [18] contains a thorough survey on CP in finite-dimensional spaces. It is worth noting that most of the existence results for CPs in infinite-dimensional spaces assume the *monotonicity* of the operator under consideration. We note that when K is a closed convex cone, Karamardian [6] has shown that VI(T,K) and CP both have the same solution set.

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In Section 4, we first give some existence results for the complementarity problem involving a completely continuous operator over an arbitrary closed convex cone in a reflexive real Banach space. In particular, several sufficient conditions for the existence of a solution to the complementarity problem are given without the monotonicity assumption of the operators. We then deal with the case when T is pseudo-monotone.

2. EXISTENCE RESULTS FOR THE VARIATIONAL INEQUALITY

Let K be a nonempty subset of B and T be an operator from K into B^*. The operator T is said to be *continuous on finite-dimensional subspaces* if T is continuous on $K \cap M$ for every finite-dimensional subspace M of B such that $K \cap M$ is nonempty. Unless specified otherwise, the topology of a Banach space mentioned in this paper refers to the norm topology.

The following theorem is the key result of this paper.

THEOREM 2.1. *Let K be a nonempty weakly compact convex set in a real Banach space B. Suppose that $T : K \to B^*$ satisfies the condition: for each $\{x_n\} \subset K$ such that $\{x_n\}$ converges weakly to x, we have*

$$\liminf_{n\to\infty}(y - x_n, T x_n) \leq (y - x, T x), \qquad y \in K.$$

Then VI(T, K) has a solution.

PROOF. For each $y \in K$, let

$$F(y) = \{x \in K \mid (y - x, T x) \geq 0\}.$$

Then $F(y)$ is nonempty since $y \in F(y)$ for each $y \in K$. Let $\{x_n\}$ be any sequence in $F(y)$. Since K is weakly compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging weakly to $\bar{x} \in K$. For each n_k, we have

$$(y - x_{n_k}, T x_{n_k}) \geq 0. \tag{2.1}$$

It follows from the assumption and (2.1) that $(y-\bar{x}, T\bar{x}) \geq 0$. Therefore, $\bar{x} \in F(y)$ and hence $F(y)$ is weakly countably compact. By Eberlein's theorem [19, p.147], $F(y)$ is weakly compact and hence weakly closed for each $y \in K$.

Now, let

$$A = \{(x, y) \in K \times K \mid (y - x, T x) \geq 0\}.$$

Then $(x, x) \in A$ for all $x \in K$. Also, by the above proof, for each $y \in K$, the set

$$\{x \in K \mid (x, y) \in A\}$$

is weakly closed. It is clear that the set $\{y \in K \mid (x, y) \notin A\}$ is empty or convex for each $x \in K$. Therefore, by [5, Lemma 4], there exists $x_0 \in K$ such that $\{x_0\} \times K \subset A$, i.e.,

$$(y - x_0, T x_0) \geq 0 \quad \text{for all} \qquad y \in K.$$

The proof is completed. ∎

It seems that Theorem 2.1 is the most natural extension of the classical Hartman-Stampacchia theorem in infinite dimensional spaces. For if B is a finite-dimensional Euclidean space and the pairing is replaced by the inner product, then the condition of Theorem 2.1 is just the direct consequence of the continuity condition. Note that there is no monotonicity assumption for the operator T in Theorem 2.1. The following result [4, Theorem 2] can be deduced from Theorem 2.1.

COROLLARY 2.2. *Let K be a nonempty weakly compact convex set in a real Banach space B. Suppose that $T : K \to B^*$ satisfies:*

(1) *for each $\{x_n\} \subset K$, weakly convergent to x,*

$$\liminf_{n\to\infty}(y, T x_n) \leq (y, T x), \quad y \in B;$$

(2) *the mapping $x \mapsto (x, T x)$ is sequentially weakly lower semicontinuous on K.*

Then VI(T, K) has a solution.

PROOF. Let $\{x_n\} \subset K$ be any sequence converging weakly to x. Then by assumptions (1) and (2), we have for each $y \in K$

$$\begin{aligned}\liminf_{n\to\infty}(y - x_n, T x_n) &\le \liminf_{n\to\infty}(y, T x_n) - \liminf_{n\to\infty}(x_n, T x_n)\\ &\le (y, T x) - (x, T x)\\ &= (y - x, T x).\end{aligned}$$

Therefore, by Theorem 2.1, the result follows. ∎

Recall that a functional $\phi : K \to \mathbf{R}$ is sequentially weakly upper semicontinuous in K if $\phi(x) \ge \limsup_{n\to\infty} \phi(x_n)$ for any sequence $\{x_n\} \subset K$ converging weakly to x. The following corollary is a direct consequence of Theorem 2.1.

COROLLART 2.3. *Let K be a nonempty weakly compact convex set in a real Banach space B. Suppose that $T : K \to B^*$ satisfies the condition: for each $y \in K$, the mapping $x \mapsto (y - x, T x)$ is sequentially weakly upper semicontinuous on K. Then VI(T, K) has a solution.*

Recall that T is *completely continuous* if T maps weakly convergent sequence to $\bar{x}$ into strongly convergent sequence to $T\bar{x}$. It is easy to see that the following corollary is also a direct consequence of Theorem 2.1.

COROLLARY 2.4. *Let K be a nonempty weakly compact convex set in a real Banach space B. If $T : K \to B^*$ is completely continuous, then VI(T, K) has a solution.*

We note that Corollary 2.4 also follows from Corollary 2.2 (see [4, Corollary 3]).

For $K \subset B$, $\mathrm{int}(K)$ and $\partial(K)$ denote the interior and boundary of K, respectively. The set $K \setminus C$ denotes the complement of C in K. For any positive real number r, C_r denotes the closed ball in B with center at zero and radius r. A subset of a Banach space is said to be *solid* if it has a nonempty interior.

Next, we give a necessary and sufficient condition for existence of a solution to VI.

THEOREM 2.5. *Let K be a nonempty closed convex set in the real reflexive Banach space B and let $T : K \to B^*$. Then $\bar{x} \in K$ is a solution of VI(T, K) if and only if there exists a solid set E such that $K \cap E$ is a bounded, closed and convex set satisfying the following conditions: $\bar{x} \in K \cap \mathrm{int}(E)$ and*

$$(x - \bar{x}, T\bar{x}) \ge 0, \qquad \text{for all} \quad x \in K \cap E. \tag{2.2}$$

PROOF. Suppose $\bar{x} \in K$ is a solution of VI(T, K). Let r be a positive number such that $\|\bar{x}\| < r$ and let $E = C_r$. Then $\bar{x} \in K \cap \mathrm{int}(E)$ and (2.2) holds.

Conversely, suppose that there exist $\bar{x} \in K$ and a solid set E satisfying the stated conditions. Let $x \in K \setminus E$ and let $0 < \lambda < 1$ be such that $\lambda x + (1 - \lambda)\bar{x} \in K \cap E$. Then, by (2.2), we have

$$\lambda(x - \bar{x}, T\bar{x}) \ge 0.$$

Thus, (1.1) holds and $\bar{x}$ is a solution of VI(T, K). ∎

Now, we give a general sufficient condition for the existence of solutions to VI(T, K).

THEOREM 2.6. *Let K be a nonempty closed convex set in the real reflexive Banach space B and let $T : K \to B^*$ satisfy the condition of Theorem 2.1. Suppose that there exists a solid set E such that $K \cap E$ is nonempty bounded closed and convex satisfying the condition that for each $x \in K \cap \partial(E)$ there exists $u \in K \cap \mathrm{int}(E)$ such that $(x - u, T x) \ge 0$. Then VI(T,K) has a solution.*

PROOF. Let $C = K \cap E$. By Theorem 2.1 there exists $\bar{x} \in C$ such that

$$(x - \bar{x}, T\bar{x}) \ge 0, \qquad \text{for all} \quad x \in C. \tag{2.3}$$

If $\bar{x} \in K \cap \mathrm{int}(E)$, $\bar{x}$ is a solution of VI(T, K) by Theorem 2.5. So, suppose that $\bar{x} \in K \cap \partial(E)$. Then by the hypothesis, there exists $u \in K \cap \mathrm{int}(E)$ such that

$$(\bar{x} - u, T\bar{x}) \ge 0.$$

It follows from (2.3) that

$$(\bar{x} - u, T\bar{x}) = 0.$$

Now, for $x \in K$ we choose $0 < \lambda < 1$ such that $\lambda x + (1 - \lambda)u \in C$. By (2.4), we have

$$\begin{aligned} 0 &\le (\lambda(x - u) + u - \bar{x}, T\bar{x}) \\ &= \lambda(x - u, T\bar{x}) \\ &= \lambda(x - \bar{x}, T\bar{x}). \end{aligned}$$

Hence, (1.1) holds and $\bar{x}$ is a solution of VI(T, K). ∎

The following theorem is a direct consequence of Theorem 2.6.

COROLLARY 2.7. *Let K and T be as those in Theorem 2.6. Suppose that there exists $r > 0$ such that for each $x \in K$ with $\|x\| = r$ there exists $u \in K$ such that $\|u\| < r$ and $(x - u, Tx) \ge 0$. Then VI(T, K) has a solution.*

We mention here some applications of Theorem 2.1.

THEOREM 2.8. *Let $T : B \to B^*$. Suppose that B is a real reflexive Banach space and that T satisfies:*

(1) *for each $\{x_n\} \subset B$ such that $\{x_n\}$ converges weakly to x,*

$$\liminf_{n\to\infty} (y - x_n, Tx_n) \le (y - x, Tx), \qquad y \in B;$$

(2) $\liminf_{\|x\|\to\infty} (x, Tx) > 0$.

Then 0 is in the range of T.

PROOF. Choose r so large that $(x, Tx) > 0$ for all $x \notin \mathrm{int}(C_r)$. By Theorem 2.1, there exists $\bar{x} \in C_r$ such that $(x - \bar{x}, T\bar{x}) \ge 0$ for all $x \in C_r$. Letting $x = 0$, we have $(\bar{x}, T\bar{x}) \le 0$. Thus $\|\bar{x}\| < r$. Consequently, $(x - \bar{x}, T\bar{x}) \ge 0$ for all $x \in B$. Therefore, $T\bar{x} = 0$. ∎

From this theorem, we have the following corollary.

COROLLARY 2.9. *If, in Theorem 2.8, the assumption (2) is replaced by the condition*

$$\lim_{\|x\|\to\infty} \frac{(x, Tx)}{\|x\|} = \infty,$$

then T is surjective, i.e., $TB = B^*$.

3. PSEUDOMONOTONE CASE

Let K be a nonempty subset of the real Banach space B, and let T be an operator from K into B^*. Following the definition of Karamardian [15], T is called *pseudo-monotone* if for any x and y in K, $(x - y, Ty) \ge 0$ implies $(x - y, Tx) \ge 0$.

We now state an existence result for VI(T, K) for the pseudo-monotone case.

THEOREM 3.1. *Let K be a closed convex set in the real reflexive Banach space B, and let $T : K \to B^*$ be a pseudo-monotone operator which is continuous on finite-dimensional subspaces. If there exists a bounded subset D of K such that for each $x \in K \setminus D$ there is $u \in D$ with $(u - x, Tx) < 0$, then VI(T, K) has a solution.*

PROOF. Let E be a closed ball in B such that $D \subset K \cap \mathrm{int}(E)$. Let $S = K \cap E$. First, we claim that there exists $\bar{x} \in S$ such that

$$(x - \bar{x}, T\bar{x}) \ge 0, \qquad \forall\, x \in S. \tag{3.1}$$

For each $x \in S$, let $S_x = \{u \in S \mid (x - u, Tx) \ge 0\}$. Since $x \in S_x$, the set S_x is nonempty for each $x \in S$. It suffices to show that the family $\{S_x \mid x \in S\}$ has the finite intersection property. Indeed, let $\{x_i \mid 1 \le i \le m\}$ be any finite subset of S and let M be the subspace of B spanned by

$\{x_i \mid 1 \leq i \leq m\}$. Let P_M be the injection of M into B and P_M^* be its adjoint. Then the operator $P_M^* T P_M$ from $S \cap M$ into M^* is continuous. Since M is finite-dimensional, we may assume, without loss of generality, that M is $\mathbb{R}^n$ for some n, where $\mathbb{R}^n$ is the n-dimensional Euclidean space. Then, by [2, Lemma 3.1], there exists $x_M \in S \cap M$ such that

$$(x - x_M,\, P_M^* T P_M x_M) \geq 0, \qquad \text{for all} \quad x \in S \cap M.$$

From this it follows that

$$(x - x_M,\, T x_M) \geq 0, \qquad \text{for all} \quad x \in S \cap M.$$

In particular, from the pseudo-monotonicity of T it follows that

$$(x_i - x_M,\, T x_i) \geq 0, \qquad \text{for} \quad i = 1, \ldots, m.$$

Hence, $x_M \in \bigcap_{i=1}^m S_{x_i}$, and thus the family $\{S_x \mid x \in S\}$ has the finite intersection property as claimed. It is also clear that the set S_x is weakly closed for each $x \in S$. Since $S_x \subset S$ for each $x \in S$ and S is weakly compact, it follows that the set $\bigcap_{x \in S} S_x$ is nonempty.

Now, let $\bar{x} \in \bigcap_{x \in S} S_x$. Then we have

$$(x - \bar{x},\, T x) \geq 0, \qquad \text{for all} \quad x \in S. \tag{3.2}$$

We claim that (3.2) implies (3.1). Indeed, for each $x \in S$ and $0 < \lambda \leq 1$, let $x_\lambda = \lambda x + (1 - \lambda)\bar{x}$. By (3.2), we have

$$(x - \bar{x},\, T x_\lambda) \geq 0, \qquad \text{for} \quad 0 < \lambda \leq 1. \tag{3.3}$$

Since $T x_\lambda \to T \bar{x}$ as $\lambda \to 0$, the assertion (3.1) follows.

By the hypothesis, we must have $\bar{x} \in D$. The result then follows from Theorem 2.5. ∎

The following result is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let K be a closed convex subset in the real reflexive Banach space B, and let $T : K \to B^*$ be a pseudo-monotone operator which is continuous on finite-dimensional subspaces. If there exists $x_0 \in K$ such that the set $D = \{x \in K \mid (x - x_0, T x_0) \leq 0\}$ is bounded, then VI(T, K) has a solution. Moreover, the solution set of VI(T, K) is convex and weakly compact.*

Proof. It follows from Theorem 3.1 that the solution set of VI(T, K) is nonempty.

Let X be the solution set of VI(T, K). To show X is convex, let x_1 and x_2 be in X and let $x_\lambda = \lambda x_1 + (1 - \lambda)x_2$, $0 \leq \lambda \leq 1$. For any $x \in K$, since T is pseudo-monotone,

$$(x - x_1,\, T x) \geq 0 \quad \text{and} \quad (x - x_2,\, T x) \geq 0.$$

Therefore,

$$(x - x_\lambda,\, T x) \geq 0, \qquad \forall x \in K.$$

By the same argument as above, we have

$$(x - x_\lambda,\, T x_\lambda) \geq 0, \qquad \forall x \in K.$$

Consequently, $x_\lambda \in X$ and hence X is convex.

Finally, to show X is weakly compact, it suffices to show X is closed since $X \subset D$. To this end, let z be any limit point of X. Then there exists a sequence $\{x_n\} \subset X$ such that $x_n \to z$. For any $x \in K$ and any n, we have $(x - x_n, T x) \geq 0$. Letting $n \to \infty$, we have $(x - z, T x) \geq 0$. The same reasoning as above gives that $z \in X$. Therefore, X is closed and the theorem is proved. ∎

4. THE COMPLEMENTARITY PROBLEM

From Theorem 2.5 and [6, Lemma 3.1], a necessary and sufficient condition for the existence of a solution to CP is given as follows.

THEOREM 4.1. *Let K be a closed convex cone in the real reflexive Banach space B and let T be an operator from K into B^*. Then $\bar{x} \in K$ is a solution of CP if and only if there exists a solid set E such that $K \cap E$ is a bounded, closed and convex subset of K satisfying the following conditions: $\bar{x} \in K \cap \operatorname{int}(E)$ and*

$$(x - \bar{x}, T\bar{x}) \geq 0, \qquad \text{for all} \quad x \in K \cap E. \tag{4.1}$$

Now, we give a general sufficient condition for the existence of a solution to the complementarity problem by Theorem 2.5 and Theorem 2.6.

THEOREM 4.2. *Let K be a closed convex cone in the real reflexive Banach space B and let T be a completely continuous operator from K into B^*. Suppose that there exists a solid set E such that $K \cap E$ is a nonempty, bounded, closed and convex subset of K satisfying the following condition: for each $x \in K \cap \partial(E)$ there exists $u \in K \cap \operatorname{int}(E)$ such that*

$$(x - u, Tx) \geq 0.$$

Then CP has a solution $\bar{x} \in K$.

The following theorem is a direct consequence of Theorem 4.2.

COROLLARY 4.3. *Let K be a closed convex cone in the real reflexive Banach space B and let T be a completely continuous operator from K into B^*. Suppose that there exists $r > 0$ such that for each $x \in K$ with $\|x\| = r$ there exists $u \in K \cap \operatorname{int}(C_r)$ with $(x - u, Tx) \geq 0$. Then CP has a solution $\bar{x} \in K$.*

Before we give some other important sufficient conditions for the existence of a solution to the CP from Theorem 4.2, let us recall some definitions. Let K be a closed convex cone of a real reflexive Banach space B. The operator T is said to be *weakly coercive* if

$$(x, Tx) \to \infty \quad \text{as} \quad \|x\| \to \infty, \qquad x \in K.$$

The operator T is said to be *coercive* if

$$\frac{(x, Tx)}{\|x\| \to \infty} \quad \text{as} \quad \|x\| \to \infty, \qquad x \in K.$$

The operator T is said to be *α-copositive* if there exists an increasing function $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$ such that

$$(x, Tx - T0) \geq \|x\| \, \alpha(\|x\|), \qquad \text{for all} \quad x \in K.$$

If $\alpha(r) = kr$ for some $k > 0$, then T is said to be *strongly copositive*.

COROLLARY 4.4. *Let K be a closed convex cone in the real reflexive Banach space B and let T be a completely continuous operator from K into B^*. Then CP has a solution $\bar{x} \in K$ under each of the following conditions:*

(1) $\liminf_{\|x\| \to \infty, x \in K} (x, Tx) > 0$;
(2) *T is weakly coercive;*
(3) *T is coercive;*
(4) *T is α-copositive;*
(5) *T is strongly copositive.*

PROOF. Suppose that the first condition holds. Then there exists an $r > 0$ such that $(x, Tx) \geq 0$ for all $x \in K$ with $\|x\| \geq r$. Therefore the result follows from Corollary 4.3 directly.

Since the condition $(i+1)$ implies the condition (i), for $i = 1, 2, 3, 4$, respectively, the corollary follows. ∎

For the pseudo-monotone case, applying Corollary 3.2 and a result of Karamardian [6, Lemma 3.1], we have the following result.

THEOREM 4.5. *Let K be a closed convex cone in the real reflexive Banach space B, and let $T : K \to B^*$ be a pseudo-monotone operator which is continuous on finite-dimensional subspaces. If there exists $x_0 \in K$ such that the set $D = \{x \in K \mid (x - x_0, Tx_0) \leq 0\}$ is bounded, then CP has a nonempty, convex and weakly compact solution set.*

A cone K is said to be *pointed* if $K \cap (-K) = \emptyset$. In case that the closed convex cone K is pointed and solid, we have the following result.

COROLLARY 4.6. *Let K be a pointed, solid, closed convex cone in the real reflexive Banach space B, and let $T : K \to B^*$ be a pseudo-monotone operator which is also continuous on finite-dimensional subspaces. If there exists $x_0 \in K$ such that $Tx_0 \in \operatorname{int}(-K^\circ)$, then CP has a nonempty, convex and weakly compact solution set.*

PROOF. If $(x_0, Tx_0) = 0$, we are done. So we may assume that $(x_0, Tx_0) > 0$. Let $D = \{x \in K \mid (x, Tx_0) \leq (x_0, Tx_0)\}$. By a result of Fan [20, Theorem 1], D is weakly compact. Hence, the result follows from Theorem 4.5.

We note that Corollary 4.6 is an infinite-dimensional generalization of [15, Theorem 4.1]. Also, a result similar to Corollary 4.6 is proved in [7, Theorem 4] with a different approach.

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